Good and Bad Functions

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We consider averaging processes (A_n) arising in ergodic theory. We say f is *good* if $(A_n f)$ converges a.e. and f is *bad* when a.e. $(A_n f)$ does not converge. We are interested in averaging processes that are not well-behaved i.e. some (or most) functions f are bad.

For example, take a standard non-atomic probability space (X, β, m) and take an increasing sequence (m_k) in $\mathbb{N}$, and consider $A_n^\tau f = \frac{1}{n} \sum_{k=1}^{n} f \circ \tau^{m_k}$ for some fixed ergodic mapping τ. There are examples of such averages and ergodic maps for which all non-constant functions $f \in L^1(X)$ are bad, indeed the averages are *fully divergent* in the sense that $\sup_{n \geq 1} |A_n^\tau f| = \infty$ a.e. if f is not bounded. With the sequence (m_k) fixed, this cannot ever occur for all ergodic maps, but when it does happen for a given map, then for any ergodic mapping, $\sup_{n \geq 1} |A_n^\tau f| = \infty$ a.e. for the generic function. It turns out that it can be quite challenging to know what functions are good or bad for a given map, and what maps are good or bad for a given function.

But a fairly complete analysis of the good and the bad is available for moving averages $\frac{1}{L_n} \sum_{k=v_n+1}^{v_n+L_n} f \circ \tau^k$. Indeed, given an ergodic map τ, every function $f \in L^1(X)$ is good for some moving average with respect to τ for which at the same time the generic function in $L^1(X)$ is bad. Also, we can characterize when a mean-zero $f \in L^1(X)$ is good for all moving averages with respect to an ergodic map τ. It occurs when the classical ergodic averages for f using τ are *completely convergent*: for all $\delta > 0$,

$$\sum_{n=1}^{\infty} m\{|\frac{1}{n} \sum_{k=1}^{n} f \circ \tau^k| \geq \delta\} < \infty.$$

This is a first category phenomenon, but the class is always larger than just the τ-coboundaries with integrable transfer function. In the polar case, if we fix the non-zero, mean-zero function f, then for a generic class of ergodic maps τ, the function is bad for some moving average with respect to τ. Nonetheless, for some classes of functions we can show the function is good for some ergodic map and all moving averages for that map. We conjecture that this is actually the case for all integrable functions, but the proof of this may rely on proving every mean-zero function is a τ-coboundary with an integrable transfer function for some ergodic map.